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avoided by the use of a second mirror fixed so as to deflect the rays from the heliostat in any desired direction. In this way a polar heliostat can be used in conjunction with either a horizontal or a vertical telescope.

19. The contents of this paper may be summarised thus :

(1) §§ 1 and 2 are introductory.

(2) §§ 3 and 4 deal with the geometry of a mirror caused to rotate with uniform angular velocity about the polar axis, which is inclined at a constant angle to the plane of the mirror.

(3) §§ 5–8 contain a discussion of the properties of a mirror similarly mounted, but rotated with such a (variable) velocity that the image of a particular star is maintained on the instrumental meridian.

(4) §§ 9–11 deal with the motion of a siderostat considered as a mirror capable of rotation about any two orthogonal axes, one of which is fixed in the plane of the instrumental meridian.

(5) § 12 describes a linkage which is capable of producing the motions required by a siderostat which is mounted equatorially.

(6) §§ 13–15 deal with the geometrical conditions on which the efficient use of a second mirror in conjunction with a cœlostat depends when the telescope is fixed in a horizontal position, and with the necessary adjustment in the position of the mirror.

(7) § 16 is devoted to an examination of certain typical positions of the telescope.

(8) §§ 17 and 18 refer to the use of a cœlostat with a telescope pointed to the zenith and to the possible advantage of a polar heliostat combined with a fixed mirror.

University Observatory, Oxford :
1905 *March* 8.

The Optical Sine-condition. By A. E. Conrady.

The remarkable theorem which forms the subject of this paper was brought to the notice of astronomers in general by the stipulations agreed upon at the Paris Congress for standardising the construction of the astrographic telescopes ; for, on the recommendation of Dr. Steinheil, one of the German delegates, it was laid down that the objectives must fulfil the sine-condition, so as to insure the formation of symmetrical images in the outer part of the relatively large field that had to be covered.

A short history of this theorem and a simple proof of it may therefore be acceptable.

Owing to its close association with the complicated defect known to opticians as "coma," it is a matter of course that this

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condition is found approximately fulfilled by all reasonably successful lens-systems, even those made by purely cut-and-try methods.

But in the case of the famous object glasses made by Fraunhofer in the early part of the last century, more particularly in the much discussed objective of the Königsberg heliometer, the fulfilment of the condition is such an exceedingly close one as to almost force one to assume that it was arrived at deliberately by computation. But the latter was most probably an elaborate trigonometrical one for an oblique pencil of rays, which, thanks to the sine-condition, modern computers can generally do without.

The first theoretical paper which gave something nearly akin to the modern form of the sine-condition was a classical one by v. Seidel,[*] in which the elementary dioptric theory was extended so as to include all terms of the third order, the result being that v. Seidel showed that a lens-system must fulfil five separate conditions in order to give a sharp and plane image of an extended object; the second of these five conditions corresponds to the modern sine-condition, but it is necessarily in an approximate algebraical form, as all trigonometrical functions were replaced by their development in series, and only the first two terms retained.

Eight years later, in 1863, R. Clausius,[†] in one of the numerous investigations which he had to undertake in defence of his second law of thermodynamics, was led to a very careful study of the concentration of radiant heat, and arrived, *inter alia*, at a differential equation which embodies the sine-condition, and the value of which for optical purposes he pointed out. To Clausius, therefore, belongs the honour of first discovery, although this has only been recognised in comparatively recent times.

The two names which are usually associated with the discovery are those of Abbe and v. Helmholtz. The former[‡] alludes to the sine-condition in one sentence of a paper published in 1873; but that has secured him priority over v. Helmholtz, as the latter himself acknowledged in a P.S. to his own paper[§] of the same year, in which the theorem was for the first time clearly stated as a separate optical one, and proved in a manner somewhat resembling that of Clausius.

Since then a very popular though not quite rigorous proof of the sine-condition, which has found its way into many books, has been supplied by Hockin.[||]

The existing proofs of this important theorem are either rather involved or else wanting in validity, inasmuch as the proof only extends to rays proceeding in a plane containing the optical axis; most of them also fail to bring out clearly the result of non-fulfilment, and none deal with the significance of the theorem in the presence of spherical aberration.

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[*] *Ast. Nach.* 1027–1029 (1855).

[†] R. Clausius, *Pogg. Ann.* cxxi. p. 1 (1864).

[‡] Abbe, M. Schultze's *Archiv für mikroskopische Anatomie*, ix. pp. 413–468 (1873).

[§] Helmholtz, *Pogg. Ann.*, Jubelband, p. 557 (1874).

[||] Hockin, *Journal Roy. Microscopical Soc.*, Ser. II., vol. iv. p. 337 (1884).

The following proof, simple as it is, seems to be free from these objections.

In fig. 1 let OZ be the optical axis of a system of lenses or mirrors, or both combined, no other restriction being imposed than that all the refracting or reflecting surfaces shall be continuous surfaces of rotation with OZ as axis, and the system therefore a "centred" one. The rays from O shall be allowed to enter the system through a narrow zone marked by the circle A with C (on OZ) as centre; after passing through the system they are assumed to emerge by the zone defined by the circle A′ with C′ as centre, and to come to a focus at O′. That all rays of such a zone must come to a common focus follows from the symmetry of the entire system round the axis OZ. For the same reason any one ray, such as O—A—A′—O′ must always remain in that plane containing the axis OZ in which it started from O; the angles β_1 and β_1' formed by the incident and emergent rays with the optical axis must be the same for all rays of the zone under consideration, and the paths of all such rays must be equal, both as to their entire length and as to corresponding constituent parts.

Owing to these properties of a centred optical system it is possible to discuss, without separate computation of the paths of rays through the system, the formation of the image of any point P near the optical axis, viz. so close to O that to any ray passing from O through the system we can find a corresponding ray from P which only forms small angles of the first order with it, and which does not become separated from it by more than small quantities of the first order anywhere within the system.

This generalisation depends on Fermat's theorem of the minimum optical path, according to which light passes from one point to another along the path which requires the least time* to traverse. For, according to the definition of a maximum or minimum, it follows at once that any path which agrees with a known optically determined path within small quantities of the *first* order is of the same length within small quantities not exceeding the *second* order, and is therefore sensibly equal to it. We are, therefore, justified in assuming that the paths traversed within the optical system by light from P are sensibly equal to those followed by light from O, and we can base our study of the image of P on a comparison of the paths outside the system.

In order that P′ may be the image of P all optical paths between P and P′ must be of equal length, for that is the

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* In certain cases the path assigned to light by the laws of refraction and reflexion is the one which requires the *longest* time. Fermat's theorem, therefore, requires an extension. We must state that light passes from point to point along a path for which the time of transit is *an extreme value*, i.e. either a minimum or a maximum.

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condition under which all light from P will arrive at P′ in the same phase of vibration so as to form a bright image.

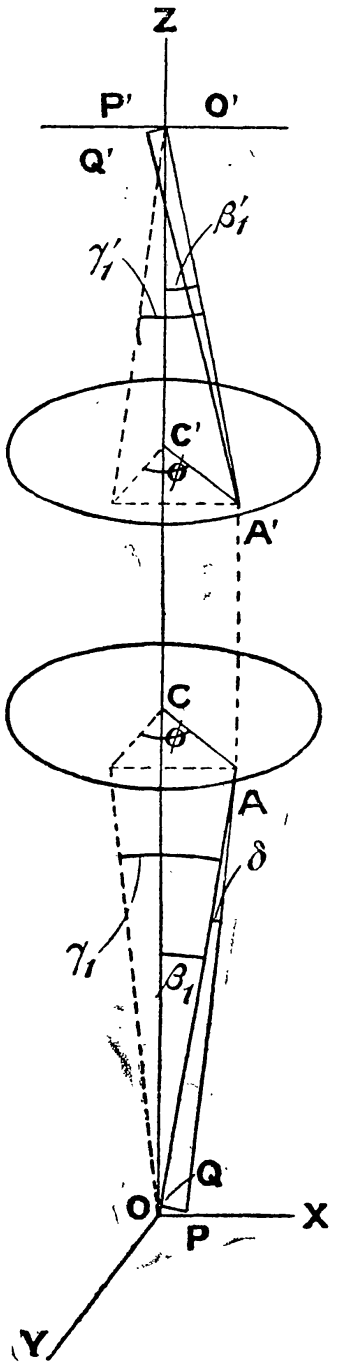

FIG. 1.

We know that all paths between O and O′—viz. the zone under consideration—*are* equal; our task, therefore, reduces itself to comparing any ray from P with its corresponding ray from O,

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and by Fermat's theorem we need not consider the portion within the system.

Now OA and PA are two such corresponding rays, and by dropping a perpendicular PQ from P upon PA we see that PA is shorter than OA by OQ. On reference to the rectangular system of coordinates shown in fig. 1, P lying on the axis OX, it is seen that, γ_1 being the angle between OA and the YZ plane, we have

$$OQ = OP \, . \, \sin \gamma_1$$

Calling the angle between radius CA and the YZ plane ϕ, we further have

$$\sin \gamma_1 = \sin \phi \sin \beta_1$$

and therefore

$$(1) \quad OQ = OP \, . \, \sin \beta_1 \, . \, \sin \phi$$

The angle ϕ being the angle between the YZ plane and the plane in which the ray OA proceeds through the system is constant for this ray; hence, by analogy, we can at once write down that the quantity P′Q′ by which A′P′ is longer than A′O′ is given by

$$(2) \quad P'Q' = O'P' \sin \beta_1' \sin \phi$$

(1) and (2) give us the *geometrical* differences between the optical paths OAA′O′ and PAA′P′. But we must compare optical paths, *i.e.* we must bear in mind that the velocity of light is inversely proportional to the refractive index, and that the geometrical paths must, therefore, be multiplied by the index of the medium in which they lie in order to be commensurable. Taking the indices surrounding object and image respectively as n and n', we therefore get the difference of the optical paths O – O′ and P – P′ given by

$$n \, . \, \overline{OQ} - n' \, . \, \overline{P'Q'} = \sin \phi \, \{ n \, . \, OP \, . \, \sin \beta_1 - n' \, . \, O'P' \sin \beta_1' \}$$

This difference varies, therefore, in general according to the value of ϕ. But it vanishes, and all optical paths between P and P′ become equal when the bracketed term becomes zero; hence a narrow zone of *any* centred optical system will yield sharp images of points not in but near the optical axis, and the position of the image is determined by the condition

$$(3) \quad n \, . \, \overline{OP} \, . \sin \beta_1 = n' \, . \, \overline{O'P'} \, . \, \sin \beta_1' \, *$$

$\dfrac{O'P'}{OP}$ is obviously the magnification of the image, for which we

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* It is interesting to note the close resemblance of this equation to the so-called optical invariant $OP \, . \, n \, . \tan \beta_1$ which is obtained in geometrical optics on the assumption of strictly collinear relation between object-space and image-space. The difference between the two proves that optical systems of wide aperture do not possess the property of collinearity.

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will introduce the symbol M. Introducing this, we arrive at the fundamental equation

$$\text{I.} \quad \mathrm{M} = \frac{n \sin \beta_{\text{I}}}{n' \sin \beta_{\text{I}}'}$$

which thus expresses a universal property of centred optical systems, whether spherically corrected or not, and which in this generality would seem to be new. Although I have not specially insisted upon it, it will be obvious that all the arguments used remain valid, not only for the case graphically shown in fig. 1 of a real inverted image, but also for virtual images, and even though the rays should cross the axis once or several times within the system.

Equation I. at once becomes the sine-condition in its usual form if, assuming a spherically corrected system, we demand that all zones of the system shall concentrate the light from P in the same point; for that is equivalent to demanding that all zones shall have the same magnifying power. Hence, taking β and β' as the general symbols for conjugate angles of convergence of any ray before and after passing through the system, we have the condition under which a system spherically corrected for the optical axis will also correctly depict points *near* the optical axis.

$$\text{II.} \quad \frac{n \sin \beta}{n' \sin \beta'} = \text{constant} = \text{magnification.}$$

Owing to the comparative uselessness of lenses which, whilst spherically corrected in the optical axis, offend against this condition, Professor Abbe has suggested that the term "aplanatism" should be defined as *including* the fulfilment of this condition, which may well be called the optical computer's best friend; for on reference to my paper on "Chromatic Correction of Object-glasses"[*] it will be seen that the sines of the angles β are habitually used to define the position of rays; hence the sine-condition enables the computer to check the usefulness of the lens he is computing by simply finding the differences of a few pairs of logarithms which he has already used in the computation for the axial point.

But, apart from its practical usefulness, the sine-condition provides a short cut in many other optical problems. The well-known rule, for instance, that the magnification of a telescope is found by dividing the clear diameter of the object-glass by the diameter of the small bright circle seen at the eyepiece-end is a necessary consequence of the sine-condition which leads to this ratio, absolutely regardless of the particular arrangement of mirrors and lenses of which the instrument may be composed.

The way in which I have deduced the condition renders it absolutely clear what must be the result of non-fulfilment: The

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[*] *Monthly Notices*, January 1904.

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different zones will have different magnifying powers; they will therefore produce images of an extra-axial point which occupy different positions on a radial line, and which together form a *line* instead of a point. Hence there can be no distinct image except in the immediate vicinity of the optical axis. In telescope-objectives the differences of magnification that are possible with the usual types are comparatively small; hence all of them answer fairly well when used *with eyepieces restricted to the optical axis.* But in microscope-objectives it is possible to produce systems which, whilst perfectly corrected for spherical and chromatic aberration in the optical axis, have differences of magnification of as much as 50 per cent. between centre and margin. Obviously this makes the formation of a recognisable image of an extra-axial point impossible, and we have therefore the remarkable possibility of lens-systems the useful field of which is of the same order as their resolving power, *i.e.* which might separate two points if placed symmetrically to the optical axis, but which would break down if the object left the optical axis *at all.*

The importance of this theorem for photographic telescopes arises from the fact which has been repeatedly proved by very laborious computations by the staff of Dr. Steinheil in Munich, that telescope-objectives which *rigorously* fulfil the sine-condition produce perfectly symmetrical images of stars up to the extreme limits of the usual astrographic field; hence the recommendation of the Paris Congress.

For telescopes the equation expressing the condition allows of a convenient modification. Telescopic objects are at a considerable distance from the instruments, so that β is a very small angle, and their angular extent rather than their linear magnitude is known. Let δ be the angle subtended by OP as seen from A; then we can replace OP in equation (3) by its equivalent OA . δ and write

$$\mathrm{OA} \,.\, \delta \,.\, n \,.\, \sin\beta = \mathrm{O'P'} \,.\, n' \,.\, \sin\beta'$$

n, the index of the medium surrounding the object, may for the telescope be put $= 1$, and OA $\sin\beta$ is evidently the semi-aperture CA which we will call y. Hence we have the sine-condition for the telescope in the form

$$\text{III.}\quad \frac{y}{n'\sin\beta'} = \frac{\mathrm{O'P'}}{\delta} = \text{constant}$$

whence it is evident that with an object at a very great distance $\dfrac{y}{n' \,.\, \sin\beta'}$ is equal to the equivalent focus of the lens-system; for it is the distance at which the image subtends the same angle as that subtended by the object. Of course in most telescopes n' is also equal to unity, the image being formed in air; but the more general form of III. includes such instruments as the one filled

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with water which Airy used in a famous experiment concerning the constant of aberration.

As to the significance of the sine-condition in systems affected with spherical aberration, it may there be used to secure symmetrical images by so proportioning the magnification of the different zones to the distance from the lens-system at which they come to a focus that all the cones have a common axis (see fig. 2). As the longitudinal aberration of lens-systems of moderate aperture—like the departure of successive zones from the sine-condition—is proportional to the second power of the aperture, this end can be completely attained in such cases. This extension of the theorem is important, as it can be used considerably to abridge the amount of computation necessary to arrive at a perfect lens, and also because in some cases spherical correction has to be sacrificed to other conditions.

This paper may fittingly be closed with some remarks which may supply the answer to a question which will occur to many, whether it is of any use making elaborate computations from which to construct object-glasses when the foremost opticians have claimed and proved that an hour's "figuring" will completely alter the correction of an object-glass. The answer is that this is quite true *so far as spherical correction in the optical axis is concerned*, but that it is wrong when chromatic correction is considered, and hopelessly so when it comes to fulfilling the sine-condition. For the process of "figuring" locally removes layers of glass the thickness of which is of the order of a wave-length of light; in fact, it requires considerable polishing to remove even that thickness; but, if done in the right place, it certainly does completely alter the spherical correction. And inasmuch as it is impossible rigorously to realise the strictly spherical surfaces assumed by the computer, it is indeed useless to carry the computation to great refinement so far as spherical correction is concerned; the result of such misplaced energy almost invariably is a spherically *under*-corrected objective owing to the tendency of the polishing tools slightly to round the extreme edges in the manner which is taken advantage of in

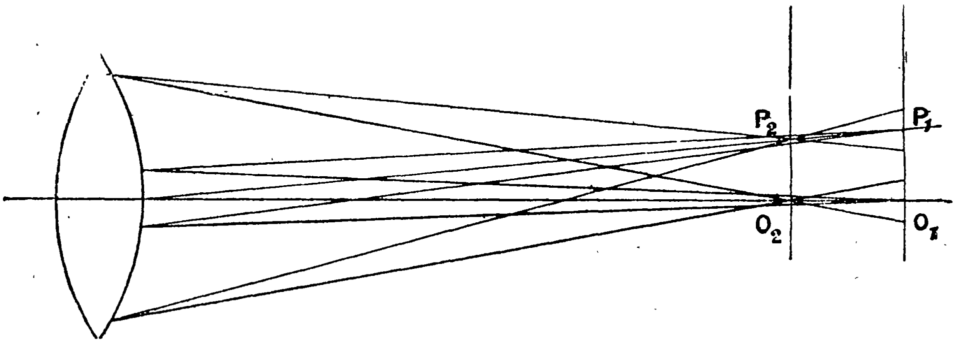

FIG. 2.

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the "figuring" of mirrors. But it is otherwise with chromatic aberration. Sensibly to change the colour for which an objective has minimum focus means a very considerable alteration, one which it would be hopeless to try and effect by *any* polishing process; it means an alteration by turning and subsequent regrinding of the tools employed, and a complete reworking of at least one surface of the objective; or a still greater alteration of two surfaces if a given focus is to be maintained. And if the sine-condition is to be fulfilled, matters become even worse.

The well-known type of object-glass, for instance, which consists of an equi-convex crown and a practically plano-concave flint, offends against the sine-condition to the extent of a difference of magnification between centre and margin, which is of the small order of one-tenth of one per cent. Yet to correct this comparatively slight error and to secure images as symmetrical as should be demanded for photographic purposes, the crown-lens will have to be altered until the curvature of its outside is about half that of the inside, whilst the flint-glass will have assumed a pronounced meniscus-form, if indeed it should be at all possible to carry out so drastic an alteration without making the glasses too thin. In fact, it may be stated, without fear of serious contradiction, that it will always be a hard task to produce by rule of thumb an objective having *minimum focus for a prescribed wave-length*, and that it would be a hopeless enterprise to try to rigorously fulfil the sine-condition without careful computation.

On the Large Sun-spot of 1905 *January* 29–*February* 11, *and Contemporaneous Magnetic Disturbances, observed at the Royal Observatory, Greenwich.*

(*Communicated by the Astronomer-Royal.*)

The largest sun-spot as yet photographed at the Royal Observatory, Greenwich, appeared at the east limb of the Sun on 1905 January 28, and passed off at the west limb on February 11. This was the rotation in which it attained its greatest dimensions; but it had been seen in the preceding rotation as well, and reappeared in the third rotation. It was first photographed at Greenwich on January 7, that day being the first occasion on which solar photographs were obtained this year. It is therefore not yet certain as to whether the group formed first in the visible hemisphere or in the one turned from us. If the latter, it would have come into view at the east limb on January 1. The group is now (March 10) completing its third rotation and passing out of sight at the west limb. The Sun was photographed at Greenwich upon five days during the first apparition of the group, upon ten days during the second,